AD NUMBER
AD200314
LIMITATION CHANGES
TO: **Approved for public release; distribution is unlimited.**
FROM: **Distribution authorized to U.S. Gov't. agencies and their contractors; Administrative/Operational Use; 23 APR 1958. Other requests shall be referred to Army Ballistic Missile Agency, Redstone Arsenal, AL.**
AUTHORITY
USAAMC ltr dtd 14 Feb 1969

UNCLASSIFIED

AD 200314

Armed Services Technical Information Agency

ARLINGTON HALL STATION
ARLINGTON 12 VIRGINIA

FOR MICRO-CARD CONTROL ONLY

1 OF 1

code 25

UNCLASSIFIED

23 April 1958

Report No. DA-TR-5-58
Copy No. A 51

ARMY BALLISTIC MISSILE AGENCY

REDSTONE ARSENAL, ALABAMA

THE MOMENT OF INERTIA OF A LIQUID IN A CIRCULAR CYLINDRICAL TANK

DEVELOPMENT OPERATIONS DIVISION

BEST

AVAILABLE

COPY

DA-TR-5-58

ARMY BALLISTIC MISSILE AGENCY
REDSTONE ARSENAL, ALABAMA
DEVELOPMENT OPERATIONS DIVISION
AEROBALLISTICS LABORATORY
DYNAMICS ANALYSIS BRANCH

ABMA TECHNICAL REPORT

23 April 1958

THE MOMENT OF INERTIA OF A LIQUID IN A CIRCULAR CYLINDRICAL TANK

SIGNATURE PAGE

ORIGINATOR:

Helmut F. Bauer

HELMUT F. BAUER
Aero Res Engineer
Vibration and Flutter Section

APPROVED:

F. W. [illegible]
Chief, Vibration & Flutter Section

HELMUT J. HORN
Chief, Dynamics Analysis Branch

for

E. D. GEISSLER
Director, Aeroballistics Laboratory

LIST OF FIGURES

Fig. 9a: Ratio of moment of inertia of fluid and solid body versus fluid height for different frequencies and damping (partially filled tank)

Fig. 9b: Phase angle of ratio of moment of inertia of fluid and solid body versus fluid height for different frequencies and damping (partially filled tank)

CONTENTS

I. SUMMARY

In previous calculations, the moment of inertia of a liquid in a circular cylindrical tank was treated as a mass-point or solid cylindrical body. Since, however, the moment of inertia of the fluid is considerably different from those, the following investigation was carried out.

The solutions of the fluid dynamics equations were derived for an ideal fluid (incompressible and nonviscous) in a completely filled circular cylindrical tank. The moment of inertia of the ideal fluid was compared with the moment of inertia of the same fluid in a frozen state. Due to the complexity of the case with friction, no damping could be incorporated. But, it can be seen that the values of the moment of inertia ratio $\frac{I_{liquid}}{I_{solid}}$ with damping are located between the ratio of the ideal fluid case and unity.

The moment of inertia of a liquid in a partially filled tank is strongly frequency-dependent. It is obtained for translational and rotational motion of the tank from the results of Reference 1. Damping was introduced as in Reference 2.

II. INTRODUCTION

To obtain a more correct value for the control frequency, in which the moment of inertia of the missile appears as I^{-1}, the following investigation was carried out.

Since the fluid in the tanks was considered as a mass point in its center of gravity in the undisturbed position of the fluid, which obviously is a very rough approximation, a better approach was made by solving the hydrodynamic equations. The moment of inertia can then be calculated with the theorem of Steiner for any pivot point, if the moments of inertia for a parallel and rotational motion around the center of gravity of the undisturbed fluid are known. In order to find these values a few additional hydrodynamic problems had to be solved for a completely enclosed fluid mass in the cylindrical tank. (full tank). The values for a partially filled tank can be taken from Reference 1. The damping effect was considered in the same way as it was taken in Reference 2.

We consider a cylindrical tank filled (completely or partially) with fluid. The motion of the fluid in the tank causes a moment around the center of gravity of the undisturbed fluid. Since the moment is

$$M = I\ddot{\vartheta} \qquad (1)$$

the moment of inertia I can be obtained. For any pivot point of a tank motion we are able to calculate with the theorem of Steiner the moment of inertia I, if we know the moments of inertia of horizontal and rotational motions. (Fig. 2)

The total moment of inertia is

$$I = m\ell^2 + I_{par} + I_{rot} \qquad (2)$$

III. NOMENCLATURE

r, ϕ, z	cylindrical coordinates (See Fig. 1)
$\Phi, \Psi, \bar{\Phi}$	velocity potentials
t	time
ρ	fluid density
b	acceleration in x-direction
p	pressure
h	fluid height
a	tank radius
ω_n	natural circular frequency of fluid
ω	circular forced frequency
x_0	small displacement in x-direction
i	imaginary unit
l	distance of pivot point to undisturbed center of gravity of fluid
θ_0	small angular displacement around center of gravity of undisturbed fluid
M	fluid moment around center of gravity of undisturbed fluid (pos. in clockwise direction)
I_ν	Bessel function of the order ν of the first kind
ε_n	zeros of the first derivative of the Bessel function of first order and first kind. $I_1'(\varepsilon_n) = 0$ $(n = 1, 2, \cdots)$
$\cdot$	means differentiation with respect to time t
[illegible]	fluid mass
[illegible]	moment of inertia

Subscript par. means: for parallel motion

Subscript rot. means: for rotational motion around y-axis

IV. COMPLETELY FILLED CYLINDRICAL TANK

We consider here a completely filled cylindrical tank , in which we do not have a free fluid surface.

(a) Translative Motion (Fig. 1)

Since we consider the fluid incompressible, we immediately see that the moment for a full tank for translative motion is zero.

$$M_{par} = 0 \tag{3}$$

The normal fluid velocities at the tank boundaries are the same as the boundary velocities normal to itself. For inviscid fluid we have to solve (See Ref. 1)

$$\Delta \Phi = 0 \tag{4}$$

with the boundary conditions

$$\frac{\partial \Phi}{\partial z} = 0 \quad \text{for} \quad z = \pm \frac{h}{2} \quad \text{at bottom and top}$$

$$\frac{\partial \Phi}{\partial r} = i\omega x_0 e^{i\omega t} \cos \phi \quad \text{for} \quad r = a \quad \text{at tank wall} \tag{5}$$

Either by physical consideration or by solving this equation with the given boundary conditions it can be found that the velocity potential is independent from the axial coordinate z and is

$$\Phi(r, \phi) = i\omega x_0 e^{i\omega t} r \cos \phi \tag{6}$$

Since the pressures at the boundaries are

$$P_{wall} = -\rho\left(\frac{\partial \Phi}{\partial t}\right)_{r=a} + \rho b\left(\frac{h}{2} - z\right) = \rho\omega^2 x_0 e^{i\omega t} a\cos\phi + \rho b\left(\frac{h}{2} - z\right) \qquad (7)$$

$$P_{top} = -\rho\left(\frac{\partial \Phi}{\partial t}\right)_{z=\frac{h}{2}} = \rho\omega^2 x_0 e^{i\omega t} r\cos\phi \qquad (8)$$

$$P_{bottom} = -\rho\left(\frac{\partial \Phi}{\partial t}\right)_{z=-\frac{h}{2}} + \rho b h = \rho\omega^2 x_0 e^{i\omega t} r\cos\phi + \rho b h \qquad (9)$$

and the moment is

$$\vec{M} = a\int_0^{2\pi}\int_{-\frac{h}{2}}^{+\frac{h}{2}} p_{wall}\, z\cos\phi\, dz\, d\phi + \int_0^{2\pi}\int_0^{a} (p_{bottom} - p_{top})\, r^2\cos\phi\, dr\, d\phi \qquad (10)$$

we finally find for the translative motion

$$M_{par} = 0 \qquad I_{par} = 0 \qquad (11)$$

(b) Rotational Motion (Fig. 1)

For an oscillation around the y - axis as shown in Fig. 1 we shall obtain the moment of inertia I_{rot}. We have to solve (See Ref. 1) the Laplace equation

$$\Delta\Phi = 0 \qquad (12)$$

with the boundary conditions

$$\frac{\partial \Phi}{\partial r} = -i\omega\theta_0 e^{i\omega t} z\cos\phi \qquad \text{for } r = a$$

$$\frac{\partial \Phi}{\partial z} = +i\omega\theta_0 e^{i\omega t} r\cos\phi \qquad \text{for } z = \pm\frac{h}{2} \qquad (13)$$

The velocity potential $\bar{\Phi} = \bar{\Phi}(r,\phi,z,t)$ is transformed into a time independent potential $\Psi(r,\phi,z)$ by

$$\bar{\Phi} = \Psi e^{i\omega t} \tag{14}$$

which fulfills the Laplace equation

$$\Delta\Psi = 0 \tag{15}$$

and has the boundary conditions

$$\begin{aligned} \frac{\partial\Psi}{\partial r} &= -i\omega\theta_0 z\cos\phi \qquad &\text{for} \quad r = a \\ \frac{\partial\Psi}{\partial z} &= +i\omega\theta_0 r\cos\phi \qquad &\text{for} \quad z = \pm\frac{h}{2} \end{aligned} \tag{16}$$

In order to make the first boundary condition homogeneous, we introduce the transformation

$$\Psi = \Phi - i\omega\theta_0 z r\cos\phi \tag{17}$$

and obtain a new set of equations

$$\Delta\Phi = 0 \tag{18}$$

$$\begin{aligned} \frac{\partial\Phi}{\partial r} &= 0 \qquad &\text{for} \quad r = a \\ \frac{\partial\Phi}{\partial z} &= 2i\omega\theta_0 r\cos\phi \qquad &\text{for} \quad z = \pm\frac{h}{2} \end{aligned} \tag{19}$$

The solution of the Laplace equation

$$\Delta\Phi = 0 \tag{20}$$

is (see Appendix)

$$\Phi(r,\phi,z) = \sum_{n=1}^{\infty}\left\{A_n\cosh(\lambda_n z) + B_n\sinh(\lambda_n z)\right\}\cdot \left\{C_n\sin(\nu\phi) + D_n\cos(\nu\phi)\right\}\cdot I_\nu(\lambda_n r) \tag{21}$$

From the boundary conditions it was found, that

$$\nu = 1 \; ; \qquad C_n = 0 \; ; \qquad B_n = \frac{4 i \omega a^2 \theta_0}{\varepsilon_n(\varepsilon_n^2 - 1)\, I_1(\varepsilon_n) \cosh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)} \; ; \qquad A_n = 0 \; ; \tag{22}$$

And finally we obtain for the velocity potential

$$\Phi(r,\phi,z,t) = -i\omega\theta_0 e^{i\omega t} a^2 \cos\phi \left\{ \frac{z}{a}\frac{r}{a} - 4\sum_{n=1} \frac{I_1\left(\varepsilon_n \frac{r}{a}\right) \sinh\left(\varepsilon_n \frac{z}{a}\right)}{\varepsilon_n(\varepsilon_n^2-1)\, I_1(\varepsilon_n) \cosh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)} \right\} \tag{23}$$

The pressure distributions are:

$$p_{wall} = -\rho\left(\frac{\partial\Phi}{\partial t}\right)_{r=a} + \rho g\left(\frac{h}{2} - z + a\theta[1-\cos\phi]\right) \tag{24a}$$

$$p_{bottom} = -\rho\left(\frac{\partial\Phi}{\partial t}\right)_{z=-\frac{h}{2}} + \rho g\left(h + \theta[a - r\cos\phi]\right) \tag{24b}$$

$$p_{top} = -\rho\left(\frac{\partial\Phi}{\partial t}\right)_{z=\frac{h}{2}} + \rho g\,\theta[a - r\cos\phi] \tag{24c}$$

which is

$$p_{wall} = -\rho\omega^2\theta_0 e^{i\omega t} a^2 \cos\phi \left\{ \frac{z}{a} - 4\sum_{n=1}^{\infty} \frac{\sinh\left(\varepsilon_n \frac{z}{a}\right)}{\varepsilon_n(\varepsilon_n^2-1) \cosh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)} \right\} + \rho g\left(\frac{h}{2} - z - a\theta\cos\phi + a\theta\right) \tag{25a}$$

$$p_{bottom} = -\rho\omega^2\theta_0 e^{i\omega t} a^2 \cos\phi \left\{ -\frac{1}{2}\frac{h}{a}\frac{r}{a} + 4\sum_{n=1}^{\infty} \frac{I_1\left(\varepsilon_n \frac{r}{a}\right) \tanh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)}{\varepsilon_n(\varepsilon_n^2-1)\, I_1(\varepsilon_n)} \right\} + \rho g\left(h - \theta r\cos\phi + a\theta\right) \tag{25b}$$

$$p_{top} = -\rho\omega^2\theta_0 e^{i\omega t} a^2 \cos\phi \left\{ \frac{1}{2}\frac{h}{a}\frac{r}{a} - 4\sum_{n=1}^{\infty} \frac{I_1\left(\varepsilon_n \frac{r}{a}\right) \tanh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)}{\varepsilon_n(\varepsilon_n^2-1)\, I_1(\varepsilon_n)} \right\} + \rho g\left(a\theta - r\theta\cos\phi\right) \tag{25c}$$

$$\left(\theta = \theta_0 e^{i\omega t}\right)$$

with formula (..., and ... Ref. ...)

$$\int_0^a r^2 I_1\left(\varepsilon_n \frac{r}{a}\right) dr = \frac{a^3}{\varepsilon_n^2} I_1(\varepsilon_n)$$

we obtain for the moment around the center of gravity

$$\tilde{M} = - m a^2 \omega^2 \theta_0 e^{i\omega t} \left\{ \frac{1}{12}\left(\frac{h}{a}\right)^2 - \frac{1}{4} + \right.$$

$$\left. + 4 \sum_{n=1}^{\infty} \frac{\dfrac{2 \tanh\left(\frac{\varepsilon_n}{2} \cdot \frac{h}{a}\right)}{\frac{\varepsilon_n}{2} \cdot \frac{h}{a}} - 1}{\varepsilon_n^2 (\varepsilon_n^2 - 1)} \right\} \qquad (26)$$

Since

$$\frac{1}{2} = 4 \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2 (\varepsilon_n^2 - 1)} .$$

the moment is then

$$\tilde{M} = - m a^2 \omega^2 \theta_0 e^{i\omega t} \left\{ \frac{1}{12}\left(\frac{h}{a}\right)^2 + \frac{1}{4} + 4 \sum_{n=1}^{\infty} \frac{\dfrac{2 \tanh\left(\frac{\varepsilon_n}{2} \frac{h}{a}\right)}{\left(\frac{\varepsilon_n}{2} \frac{h}{a}\right)} - 2}{\varepsilon_n^2 (\varepsilon_n^2 - 1)} \right\} \qquad (27)$$

Since

$$\dot{M} = I \ddot{\theta} = - \omega^2 \theta I$$

the moment of inertia for a filled cylinder with incompressible and non-viscous fluid is

$$I = m a^2 \left\{ \frac{1}{12}\left(\frac{h}{a}\right)^2 + \frac{1}{4} + 4 \sum_{n=1}^{\infty} \frac{\dfrac{2 \tanh\left(\frac{\varepsilon_n}{2} \frac{h}{a}\right)}{\left(\frac{\varepsilon_n}{2} \frac{h}{a}\right)} - 2}{\varepsilon_n^2 (\varepsilon_n^2 - 1)} \right\} \qquad (28)$$

where

$$I_0 = m a^2 \left\{ \frac{1}{12}\left(\frac{h}{a}\right)^2 + \frac{1}{4} \right\} \qquad (29a)$$

is the moment of inertia of a rigid cylinder with height h and radius a and

$$I_{corr} = 4 m a^2 \sum_{n=1}^{\infty} \frac{\dfrac{2 \tanh\left(\frac{\varepsilon_n}{2} \frac{h}{a}\right)}{\left(\frac{\varepsilon_n}{2} \frac{h}{a}\right)} - 2}{\varepsilon_n^2 (\varepsilon_n^2 - 1)} \qquad (29b)$$

is the correction moment of inertia due to the fact that the fluid in the cylinder is not rigid. (Fig. 3) In Fig. 4 the ratio of the moment of inertia of the fluid and moment of inertia of the "frozen state" is shown.

$$\frac{I_{fluid}}{I_{rigid}} = 1 + 48\sum_{n=1}^{\infty}\left\{\frac{\frac{2\tanh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)}{\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)\left(\left(\frac{h}{a}\right)^2+3\right)} - \frac{1}{\left(\frac{h}{a}\right)^2+3}}{\varepsilon_n^2\left(\varepsilon_n^2-1\right)}\right\} - \frac{6}{\left(\frac{h}{a}\right)^2+3} \tag{29c}$$

For $\frac{h}{a} \rightarrow \infty$ we obtain

$$\lim_{\frac{h}{a}\rightarrow\infty}\left(\frac{I_{fluid}}{I_{rigid}}\right) = 1$$

and for $\frac{h}{a} \rightarrow 0$

$$\lim_{\frac{h}{a}\rightarrow 0}\left(\frac{I_{fluid}}{I_{rigid}}\right) = 1$$

V. PARTIALLY FILLED CYLINDRICAL TANK

For a partially filled tank the moments have been obtained in Ref. 1 and can be taken from there.

(a) Translative Motion (Fig. 1)

The moment of a translative motion referred to the undisturbed position of the center of gravity of the fluid is (See 57 - Ref. 2)

$$\vec{M} = \omega^2 x_0 e^{i\omega t} m a\left[\frac{1}{4\frac{h}{a}} + \sum_{n=1}^{\infty}\frac{\tanh\left(\varepsilon_n\frac{h}{a}\right) + \frac{4}{\left(\varepsilon_n\frac{h}{a}\right)\cosh\left(\varepsilon_n\frac{h}{a}\right)} - \frac{2}{\varepsilon_n\frac{h}{a}}}{\varepsilon_n\left(\varepsilon_n^2-1\right)\left(\frac{\omega_n^2}{\omega^2}-1\right)}\right] \tag{30}$$

or with

$$\sum_{n=1}^{\infty}\frac{1}{\varepsilon_n^2\left(\varepsilon_n^2-1\right)} = \frac{1}{8}$$

$$\vec{M} = \omega^2 x_0 e^{i\omega t} m a\sum_{n=1}^{\infty}\frac{1}{\varepsilon_n^2\left(\varepsilon_n^2-1\right)}\left\{\frac{\frac{2}{\left(\frac{h}{a}\right)}\left(\frac{\omega_n^2}{\omega^2}-2\right) + \varepsilon_n\tanh\left(\varepsilon_n\frac{h}{a}\right) + \frac{4}{\frac{h}{a}\cosh\left(\varepsilon_n\frac{h}{a}\right)}}{\left(\frac{\omega_n^2}{\omega^2}-1\right)}\right\} \tag{31}$$

Introducing a damping factor in the way we did in Ref. 2., we obtain for the moment

$$\vec{M} = \omega^2 x_0 e^{i\omega t} m a\sum_{n=1}^{\infty}\frac{\frac{2}{\frac{h}{a}}\left(\frac{\omega_n^2}{\omega^2}-2\right) + \varepsilon_n\tanh\left(\varepsilon_n\frac{h}{a}\right) + \frac{4}{\frac{h}{a}\cosh\left(\varepsilon_n\frac{h}{a}\right)}}{\varepsilon_n^2\left(\varepsilon_n^2-1\right)\left(\frac{\omega_n^2}{\omega^2}-1+ig\frac{\omega_n}{\omega}\right)} \tag{32}$$

and finally for the moment of inertia ($X_0 = l\theta_0$)

$$\frac{I_{eff}}{\cos} = mal \sum_{n=1}^{\infty} \frac{\frac{2}{\frac{h}{a}}\left(\frac{\omega_n^2}{\omega^2}-2\right) + \varepsilon_n \tanh\left(\varepsilon_n \frac{h}{a}\right) + \frac{4}{\frac{h}{a}\cosh\left(\varepsilon_n \frac{h}{a}\right)}}{\varepsilon_n^2(\varepsilon_n^2-1)\left(\frac{\omega_n^2}{\omega^2}-1+ig\frac{\omega_n}{\omega}\right)} = \chi_1 mal \qquad (33)$$

$$\frac{I_{eff}}{mal} = \sum_{n=1}^{\infty} \frac{\frac{2}{\frac{h}{a}}\left(\frac{\omega_n^2}{\omega^2}-2\right) + \varepsilon_n \tanh\left(\varepsilon_n \frac{h}{a}\right) + \frac{4}{\frac{h}{a}\cosh\left(\varepsilon_n \frac{h}{a}\right)}}{\varepsilon_n^2(\varepsilon_n^2-1)\left(\frac{\omega_n^2}{\omega^2}-1+ig\frac{\omega_n}{\omega}\right)} \quad \text{(Fig. 5, 6)} \qquad (33a)$$

(b) <u>Rotational Motion</u> (Fig. 1)

The moment referred to the undisturbed position of the center of gravity of the fluid for a partially filled cylindrical tank performing a rotational motion around the y - axis as shown in Fig. 1 is (See (66) Ref. 1)

$$M = -\omega^2\theta_0 e^{i\omega t} ma^2 \Big[\frac{1}{12}\left(\frac{h}{a}\right)^2 - \frac{1}{8} + \frac{g}{a\omega^2}\frac{h}{2a} + 2\sum_{n=1}^{\infty} \frac{1}{\varepsilon_n(\varepsilon_n^2-1)\left(\frac{\omega_n^2}{\omega^2}-1\right)} \Big\{ \left(\frac{2}{\varepsilon_n} - \frac{4g}{a\omega^2}\frac{1}{\varepsilon_n \frac{h}{a}}\right)\frac{1}{\cosh\left(\varepsilon_n \frac{h}{a}\right)} + \left(\frac{4g}{a\omega^2}\frac{1}{\varepsilon_n\frac{h}{a}} + \frac{1}{2\varepsilon_n}\right) + \left(\frac{h}{2a} - \frac{g}{a\omega^2} - \frac{4}{\varepsilon_n}\frac{1}{\varepsilon_n \frac{h}{a}}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right)\Big\}\Big] \qquad (34)$$

Since $\omega_n^2 = \frac{g}{a}\varepsilon_n \tanh\left(\varepsilon_n \frac{h}{a}\right) \quad (n = 1, 2, 3, \ldots)$

and $\frac{1}{8} = \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2(\varepsilon_n^2-1)}$ (Ref. 1)

We finally obtain by introducing a damping factor

$$M = -\omega^2\theta_0 e^{i\omega t} ma^2 \Big[\frac{1}{12}\left(\frac{h}{a}\right)^2 + \frac{1}{4} + 2\sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2(\varepsilon_n^2-1)\left(\frac{\omega_n^2}{\omega^2}-1+ig\frac{\omega_n}{\omega}\right)} \cdot \Big\{ \frac{2}{\cosh\left(\varepsilon_n \frac{h}{a}\right)} - \frac{4}{\varepsilon_n \frac{h}{a}}\frac{\frac{\omega_n^2}{\omega^2}}{\sinh\left(\varepsilon_n \frac{h}{a}\right)} + \frac{4}{\varepsilon_n \frac{h}{a}}\frac{\frac{\omega_n^2}{\omega^2}}{\tanh\left(\varepsilon_n \frac{h}{a}\right)} + 2 + \frac{1}{4}\left(\varepsilon_n \frac{h}{a}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right) - 3\frac{\omega_n^2}{\omega^2} - \frac{4}{\varepsilon_n \frac{h}{a}}\tanh\left(\varepsilon_n \frac{h}{a}\right) + \frac{\omega_n^2}{\omega^2}\left(\frac{[illegible]}{[illegible]\tanh\left(\varepsilon_n \frac{h}{a}\right)}\right)\Big\}\Big] \qquad (35)$$

and finally for the moment of inertia

$$I_{rot} = I_0 + I_{corr}^{(rot)}$$

with $I_0 = ma^2\left[\frac{1}{12}\left(\frac{h}{a}\right)^2 + \frac{1}{4}\right]$ (36a)

$$I_{corr}^{(rot)} = 2ma^2 \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2(\varepsilon_n^2-1)\left(\frac{\omega_n^2}{\omega^2} - 1 + ig\frac{\omega_n}{\omega}\right)} \left\{ \frac{2}{\cosh(\varepsilon_n \frac{h}{a})} - \frac{4}{\varepsilon_n \frac{h}{a}} \cdot \frac{\frac{\omega_n^2}{\omega^2}}{\sinh(\varepsilon_n \frac{h}{a})} + \frac{4}{\varepsilon_n \frac{h}{a}} \cdot \frac{\frac{\omega_n^2}{\omega^2}}{\tanh(\varepsilon_n \frac{h}{a})} + \right.$$

$$\left. + 2 + \frac{1}{4}\left(\varepsilon_n \frac{h}{a}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right) - 3\frac{\omega_n^2}{\omega^2} - \frac{4}{\varepsilon_n \frac{h}{a}}\tanh\left(\varepsilon_n \frac{h}{a}\right) + \left(\frac{\omega_n^2}{\omega^2}\right)^2 \frac{1}{\left(\varepsilon_n \frac{h}{a}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right)} \right\} \quad (36b)$$

$$\frac{I_0}{ma^2} = \frac{1}{12}\left(\frac{h}{a}\right)^2 + \frac{1}{4}$$

$$\frac{I_{corr}^{(rot)}}{ma^2} = 2 \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2(\varepsilon_n^2-1)\left(\frac{\omega_n^2}{\omega^2} - 1 + ig\frac{\omega_n}{\omega}\right)} \left\{ \frac{2}{\cosh(\varepsilon_n \frac{h}{a})} - \frac{4}{\varepsilon_n \frac{h}{a}} \cdot \frac{\frac{\omega_n^2}{\omega^2}}{\sinh(\varepsilon_n \frac{h}{a})} + \frac{4}{\varepsilon_n \frac{h}{a}} \cdot \frac{\frac{\omega_n^2}{\omega^2}}{\tanh(\varepsilon_n \frac{h}{a})} + \right. \quad (36c)$$

$$\left. + 2 + \frac{1}{4}\left(\varepsilon_n \frac{h}{a}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right) - 3\frac{\omega_n^2}{\omega^2} - \frac{4}{\varepsilon_n \frac{h}{a}}\tanh\left(\varepsilon_n \frac{h}{a}\right) + \left(\frac{\omega_n^2}{\omega^2}\right)^2 \frac{1}{\left(\varepsilon_n \frac{h}{a}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right)} \right\} \equiv \chi_L$$

Furthermore we obtain for the ratio of the moment of inertia of the fluid and the moment of inertia of the "frozen state"

$$\frac{I_{fluid}}{I_{rigid}} = 1 + \frac{24}{\left(\frac{h}{a}\right)^2 + 3} \sum_{n=1}^{\infty} \frac{1}{\varepsilon_n^2(\varepsilon_n^2-1)\left(\frac{\omega_n^2}{\omega^2} - 1 + ig\frac{\omega_n}{\omega}\right)} \left\{ \frac{2}{\cosh(\varepsilon_n \frac{h}{a})} - \frac{4\frac{\omega_n^2}{\omega^2}}{\left(\varepsilon_n \frac{h}{a}\right)\sinh\left(\varepsilon_n \frac{h}{a}\right)} \right. \quad (36d)$$

$$+ \frac{4}{\varepsilon_n \frac{h}{a}} \frac{\frac{\omega_n^2}{\omega^2}}{\tanh\left(\varepsilon_n \frac{h}{a}\right)} + 2 + \frac{1}{4}\left(\varepsilon_n \frac{h}{a}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right) - 3\frac{\omega_n^2}{\omega^2} - \frac{4}{\varepsilon_n \frac{h}{a}}\tanh\left(\varepsilon_n \frac{h}{a}\right) +$$

$$\left. + \left(\frac{\omega_n^2}{\omega^2}\right)^2 \frac{1}{\left(\varepsilon_n \frac{h}{a}\right)\tanh\left(\varepsilon_n \frac{h}{a}\right)} \right\}$$

which is shown in Fig. 9.

VI. APPLICATION

In a completely filled cylinder the moment of inertia for a rigid body was graphed and the correction moment of inertia due to the fact that the fluid in the cylinder is ideal was calculated and graphed. The value

$$4\sum_{n=1}^{\infty}\left[\frac{\dfrac{2\tanh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)}{\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)}-1}{\varepsilon_n^2(\varepsilon_n^2-1)}\right]-\frac{1}{2}=\frac{I^{(rel)}_{corr}}{ma^2}$$

tends for $\frac{h}{a} \to 0$ to

$$\lim_{\frac{h}{a}\to 0}\left(\frac{I_{corr}}{ma^2}\right)=0$$

Since

$$\lim_{\frac{h}{a}\to 0}\frac{\tanh\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)}{\left(\frac{\varepsilon_n}{2}\frac{h}{a}\right)}=1$$

And $$4\sum_{n=1}^{\infty}\frac{1}{\varepsilon_n^2(\varepsilon_n^2-1)}=\frac{1}{2}$$

For $\frac{h}{a} \to \infty$, we obtain

$$\lim_{\frac{h}{a}\to\infty}\left(\frac{I^{(rel)}_{corr}}{ma^2}\right)=-1$$

Fig. 3 shows the moments of inertia and the correction term.

The correction moment of inertia for a fluid mass with a free fluid surface due to translative and rotational motion was calculated and graphed for the value

$$\frac{h}{a} = 10.0$$

Different damping values g

($g = 0; 0.02; 0.05; 0.1; 0.2; 0.5; 1.0$)

and various frequencies

$$\omega^2 = 0.1\,\omega_1^2$$

$$\omega^2 = 0.5\,\omega_1^2$$

$$\omega^2 = 0.9\,\omega_1^2$$

$$\omega^2 = 1.1\,\omega_1^2$$

$$\omega^2 = 0.5\,\omega_2^2$$

(See Fig.5,6 and Fig. 7,8,9)

VII. APPENDIX

The solution of the Laplace equation

$$\Delta \Phi = 0$$

was found as in Ref. 1

$$\Phi_n = \left[A_n \cosh(\lambda_n z) + B_n \sinh(\lambda_n z)\right]\left[C_n \sin\nu\varphi + D_n \cos\nu\varphi\right] I_\nu(\lambda_n r)$$

with the boundary condition

$$\frac{\partial \Phi}{\partial r} = 0 \quad \text{for} \quad r = a, \text{ we obtain} \qquad I_\nu'(a\lambda_n) = 0$$

which is fulfilled for

$$\lambda_n = \frac{\zeta_n^{(\nu)}}{a} \qquad (n = 1, 2, 3, \ldots)$$

where the $\zeta_n^{(\nu)}$ are the zeros of I_ν'

The remaining two boundary conditions lead to the two equations

$$\frac{\partial \Phi}{\partial z} = 2i\omega\theta_0 r\cos\phi \qquad \text{for } z = \pm\frac{h}{2}$$

$$\sum_{n=1}^{\infty}\left[\frac{\xi_n}{a}A_n\sinh\left(\frac{\xi_n}{2}\frac{h}{a}\right)+B_n\frac{\xi_n}{a}\cosh\left(\frac{\xi_n}{2}\frac{h}{a}\right)\right]\left[C_n\sin\nu\phi+D_n\cos\nu\phi\right]I_\nu\left(r\frac{\xi_n^{(\nu)}}{a}\right)=$$

$$= 4i\omega\theta_0 a\sum_{n=1}^{\infty}\frac{I_1\left(\xi_n\frac{r}{a}\right)}{(\xi_n^2-1)I_1(\xi_n)}\cos\phi$$

$$\sum_{n=1}^{\infty}\left[-\frac{\xi_n}{a}A_n\sinh\left(\frac{\xi_n}{2}\frac{h}{a}\right)+B_n\frac{\xi_n}{a}\cosh\left(\frac{\xi_n}{2}\frac{h}{a}\right)\right]\left[C_n\sin\nu\phi+D_n\cos\nu\phi\right]I_\nu\left(r\frac{\xi_n^{(\nu)}}{a}\right)=$$

$$= 4i\omega\theta_0 a\sum_{n=1}^{\infty}\frac{I_1\left(\xi_n\frac{r}{a}\right)}{(\xi_n^2-1)I_1(\xi_n)}\cos\phi$$

from which we immediately can conclude, that

$C_n = 0$ [illegible] $\nu = 1$

$\xi_n^{(1)} = \xi_n$ [illegible] $D_1 = 1$, we obtain two linear equations for A_n and B_n

$$A_n\sinh\left(\frac{\xi_n}{2}\frac{h}{a}\right)+B_n\cosh\left(\frac{\xi_n}{2}\frac{h}{a}\right) = \frac{4i\omega\theta_0 a^2}{\xi_n(\xi_n^2-1)I_1(\xi_n)}$$

$$A_n\sinh\left(\frac{\xi_n}{2}\frac{h}{a}\right)-B_n\cosh\left(\frac{\xi_n}{2}\frac{h}{a}\right) = -\frac{4i\omega\theta_0 a^2}{\xi_n(\xi_n^2-1)I_1(\xi_n)}$$

and by solving these equations we obtain for the integration constants A_n and B_n

$$A_n = 0$$

$$B_n = \frac{4i\omega\theta_0 a^2}{\xi_n(\xi_n^2-1)I_1(\xi_n)\cosh\left(\frac{\xi_n}{2}\frac{h}{a}\right)}$$

Introducing this into Φ , we finally obtain

$$\Phi(r,\phi,z) = i\omega\theta_0 e^{i\omega t} a^2 \cos\phi \sum_{n=1}^{\infty} \frac{I_1(\varepsilon_n \frac{r}{a})}{\varepsilon_n(\varepsilon_n^2-1)I_1'(\varepsilon_n)} \cdot \left[\frac{\sinh(\varepsilon_n \frac{z}{a})}{\cosh(\frac{\varepsilon_n h}{2a})}\right].$$

and by applying (17) and (24) the velocity potential:

$$\Phi(r,\phi,z,t) = -i\omega\theta_0 e^{i\omega t} a^2 \cos\phi \left[\left(\frac{z}{a}\right)\left(\frac{r}{a}\right) - 4\sum_{n=1}^{\infty} \frac{I_1(\varepsilon_n \frac{r}{a})}{\varepsilon_n(\varepsilon_n^2-1)I_1'(\varepsilon_n)} \cdot \frac{\sinh(\varepsilon_n \frac{z}{a})}{\cosh(\frac{\varepsilon_n h}{2a})}\right]$$

EXAMPLE

For a tank of radius

$a = 100$ cm

and a fluid height

$h = 200$ cm

filled with liquid of density

$\rho = 1\ gr\,cm^{-3}$

The moment of inertia was calculated for a partially filled tank.

Acceleration $g = 1000 cm/sec^2$

$l = 200$ cm

Maximum rotation angle $\theta_0 = 3^\circ (0.0523)$

$l\theta_0 = 10.46$ cm

forced frequency

$f = 0.5\ sec^{-1}$

$\omega = 3.14\ sec^{-1}$

$\frac{h}{a} = 2.0$

fluid mass $m = \pi a^2 h \rho = 6.28 \cdot 10^5$ gr.

We obtain

$$I = m\left[l^2 + a l \gamma_1 + \frac{1}{12}h^2 + \frac{1}{4}a^2 + a^2\gamma_2\right]$$

$$I = 30.6 \cdot 10^{10}\ gr\,cm^2$$

The value, which considers only a masspoint for the fluid in the center of gravity of the undisturbed fluid, is

$$\bar{I} = m l^2 = 25.1 \cdot 10^{10} \text{ gr cm}^2$$

The error is about 20%, but is of course considerably higher in resonance, which is for this fluid height and acceleration at about $f = 0.62$ sec.$^{-1}$.

VIII. CONCLUSION

The moment of inertia of a liquid in a circular cylindrical tank can be considerably different from the moment of inertia of the same fluid in a "frozen state".

The fluid dynamics equations were solved for an incompressible and nonviscous liquid in a completely filled tank. The moment of inertia ratio of the ideal fluid and the same fluid in a "frozen state" was graphed versus the fluid height ratio h/a. It is seen that for very small and very high fluid heights the value approaches 1. The minimum value is approximately at h/a = 1.72. For a tank with a square base, the minimum value would be at h/a = 2.0, i.e. the maximum amount of fluid (sphere in the center) is not affected by the rotation around the center of gravity. In a circular cylindrical tank, the minimum value of the moment of inertia ratio is at a height ratio somewhat smaller than 2.0 because of the circular form. (See Fig. 3 and 4)

The moment of inertia for a partially filled tank performing translational motions was obtained from Reference 1, and was graphed for different damping values g (Reference 2) versus fluid height ratio h/a (Fig. 5 - 8). It can be seen that considerable differences occur compared with the moment of inertia of the same fluid in a "frozen state". The forced frequency plays an important role which can be seen from the graphs.

In Fig. 9, the ratio of the moment of inertia of the fluid and the moment of inertia of the fluid in a "frozen state" is graphed versus the fluid height ratio h/a for different damping values and different forced frequencies for a partially filled tank. For very low forced frequencies, it can be seen that the moment of inertia ratio is decreasing very rapidly from unity at the fluid height ratio zero to small values for small fluid

height ratios. At a fluid height about equal to the diameter, the ratio is gradually increasing to 1, and the damping effect is of less importance because of the greater fluid mass that takes part in the oscillation. For increasing forced frequencies, the curves spread out farther. Near resonance, for fluid heights less than the diameter, the value of the moment of inertia ratio changes considerably for different damping values. These theoretical values, of course, will be smaller since the fluid surface breaks up. For higher forced frequencies, the moment of inertia ratio will be greater than 1 for small fluid heights since the ripples on the fluid surface are of the magnitude of the fluid height, while for increasing fluid height, the ratio is smaller than 1 and increases very slowly to the value 1. This is due to the fact that the small fluid surface deflection adds only a very small amount to the moment of inertia.

IX. REFERENCES

1. Helmut F. Bauer: "Fluid Oscillations in a Cylindrical Tank". ABMA Report DA-TR-1-58

2. Helmut F. Bauer: "Fluid Oscillations in Cylindrical Tanks with Damping". ABMA Report DA-TR-4-58

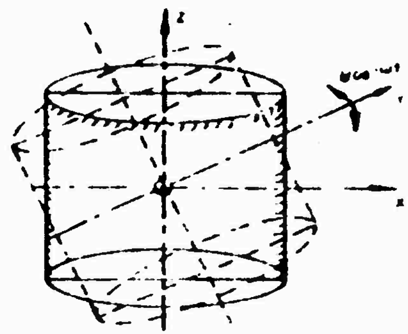


FIG 1a

FULL CYLINDRICAL TANK

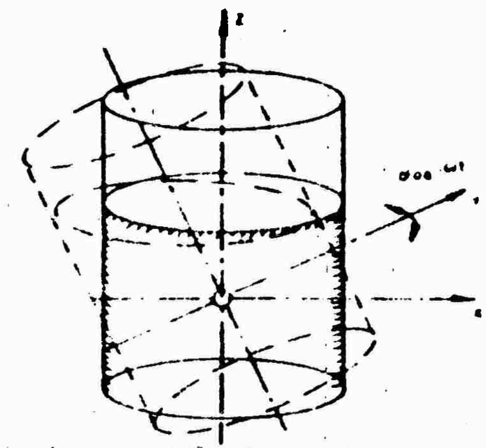


FIG 1b

PARTIALLY FILLED CYLINDRICAL TANK

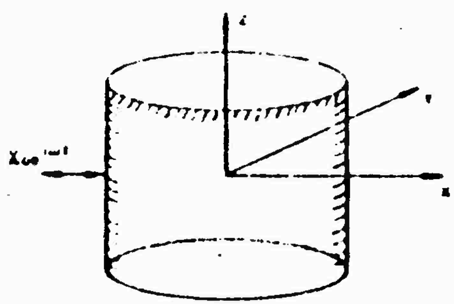


FIG 1c

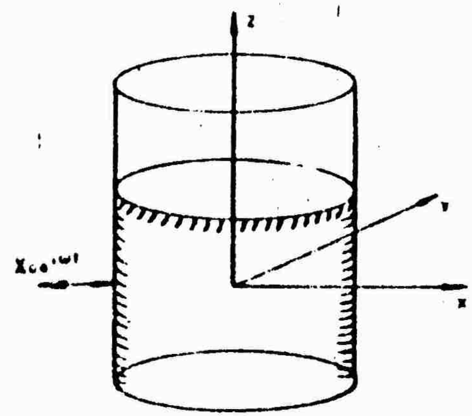


FIG 1d

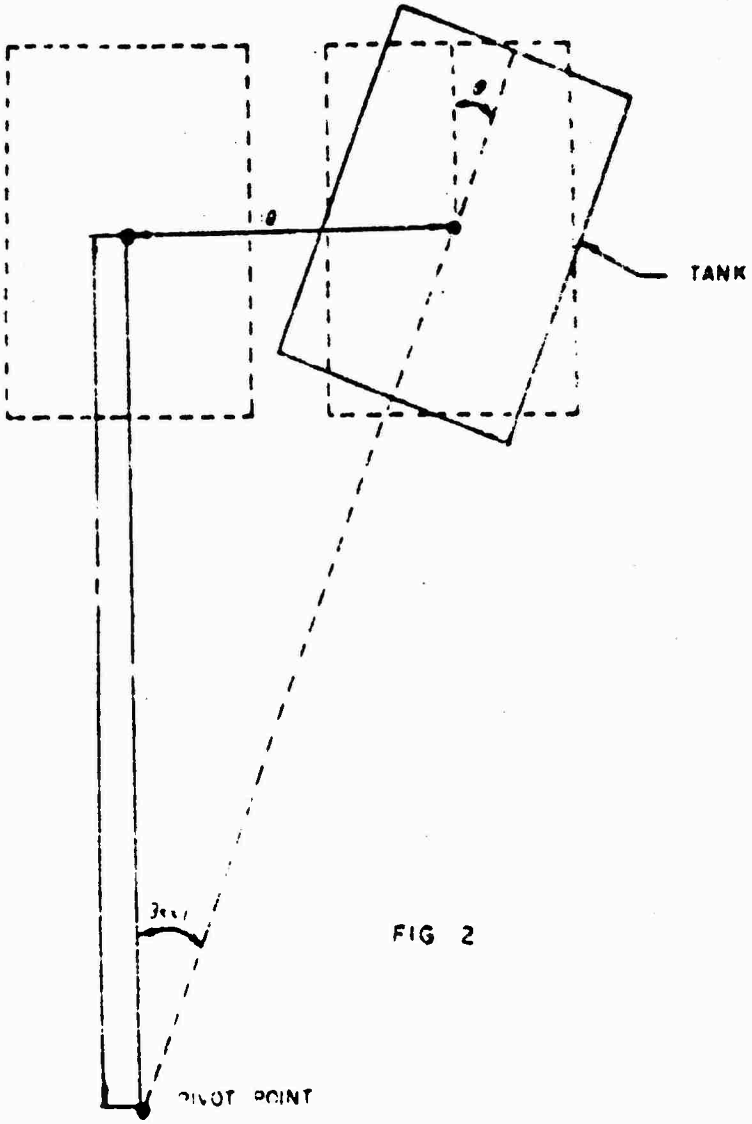


FIG 2

HORIZONTAL MOTION AMPLITUDE IS $x_0 \sin \omega t$

ROTATIONAL MOTION AMPLITUDE IS $\theta \sin \omega t$

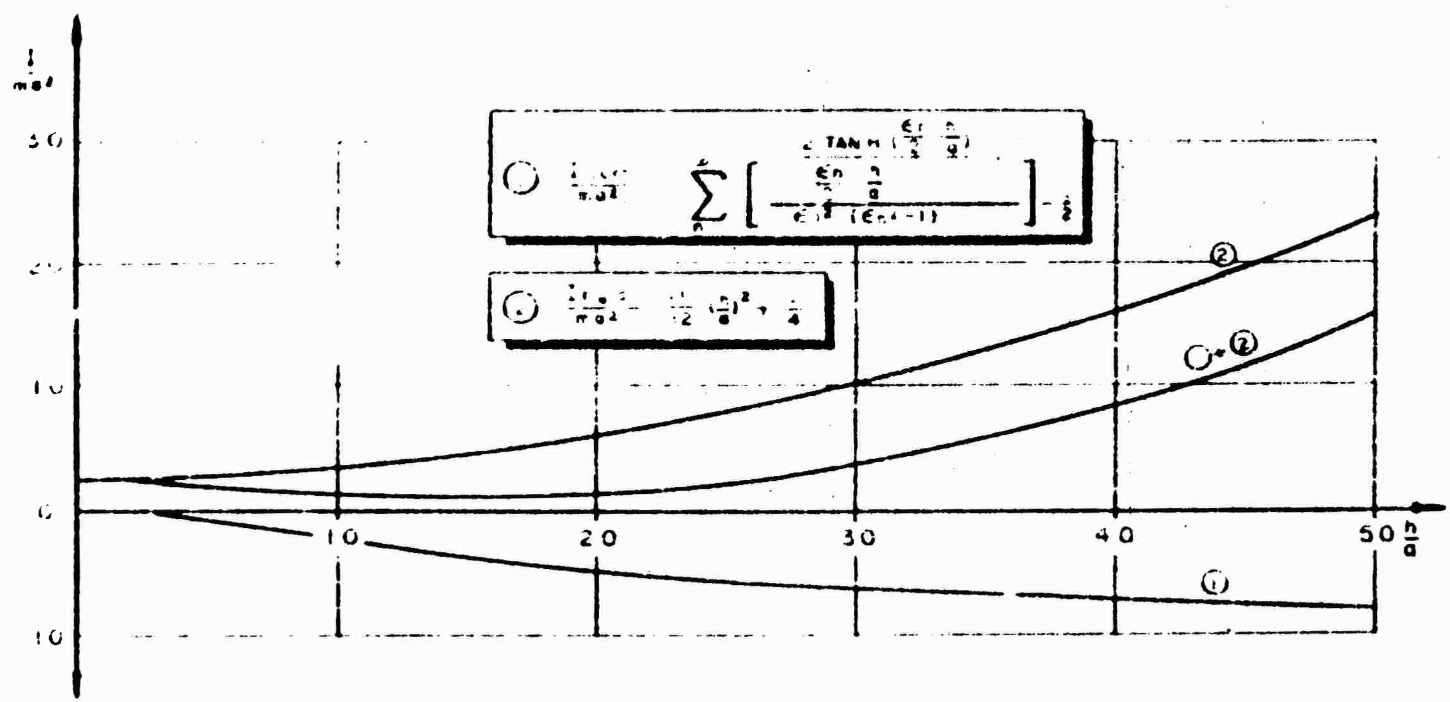


FIG 3 MOMENT OF INERTIA FOR IDEAL FLUID AND RIGID BODY
(COMPLETELY FILLED TANK)

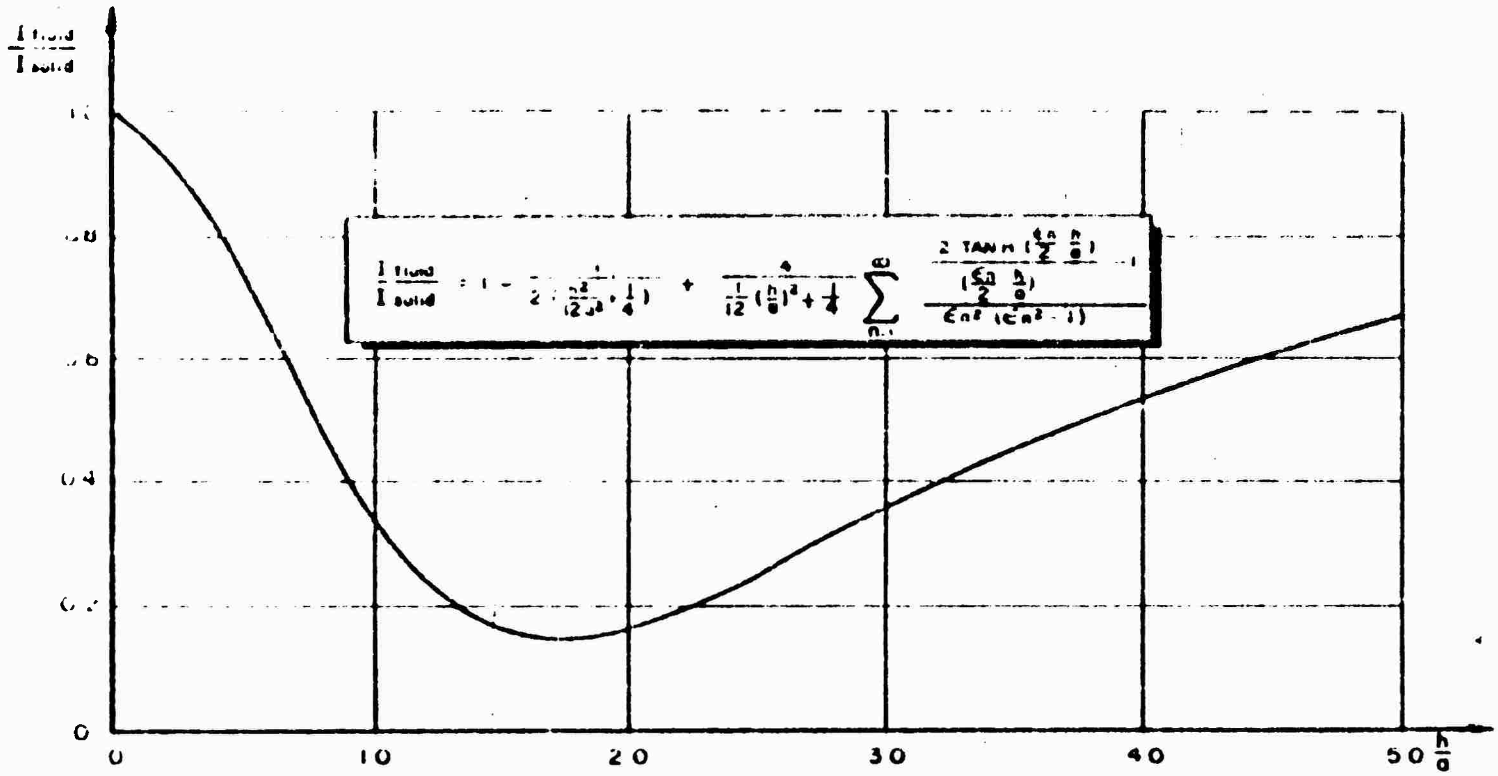
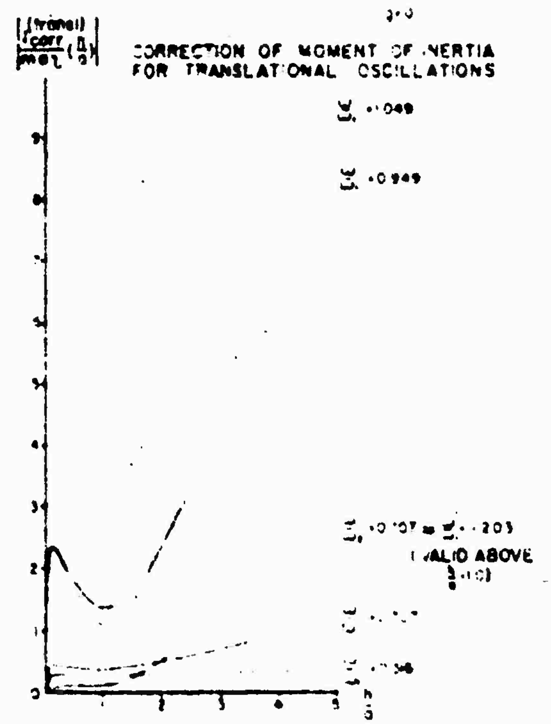


FIG 4 RATIO OF MOMENT OF INERTIA VERSUS FLUID HEIGHT RATIO $\frac{h}{a}$
(COMPLETELY FILLED TANK)


CORRECTION OF MOMENT OF INERTIA
FOR TRANSLATIONAL OSCILLATIONS
(VALID ABOVE

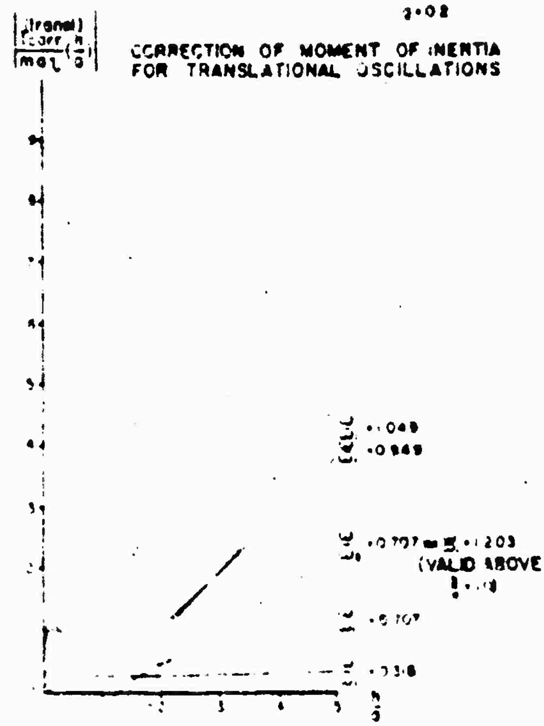
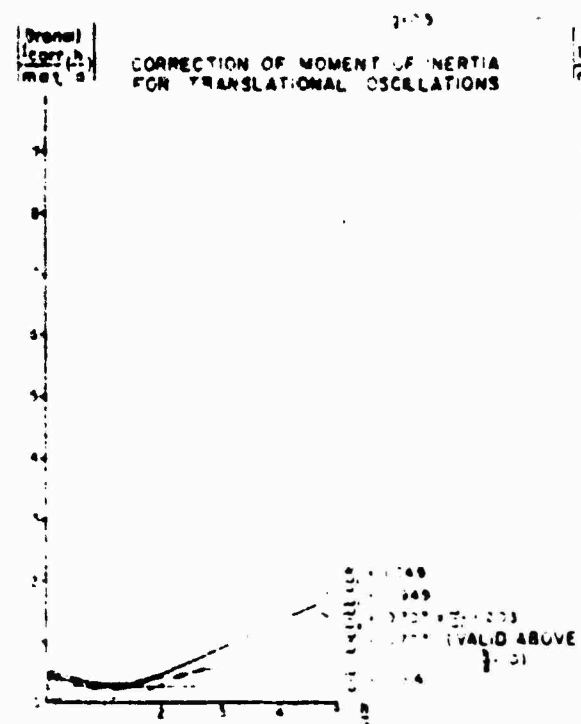
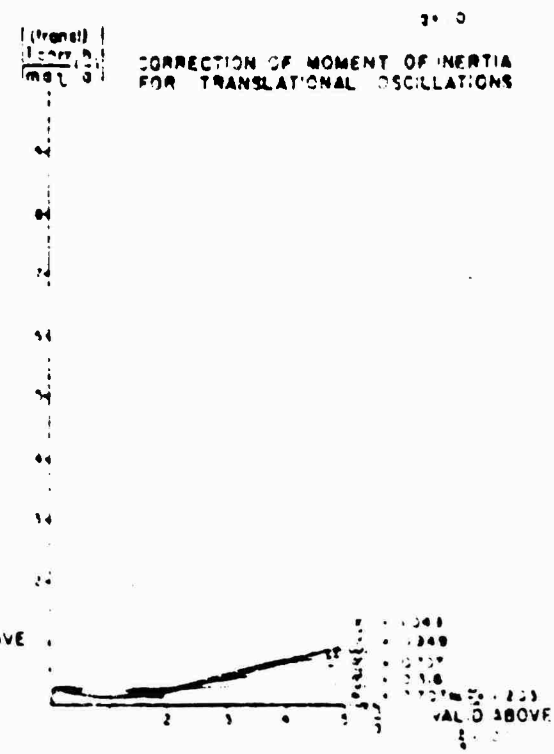
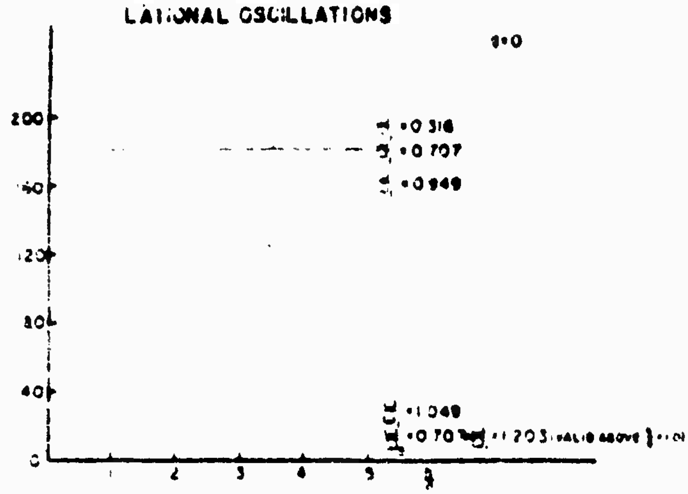

CORRECTION OF MOMENT OF INERTIA
FOR TRANSLATIONAL OSCILLATIONS
(VALID ABOVE

FIG. 5(d)


CORRECTION OF MOMENT OF INERTIA
FOR TRANSLATIONAL OSCILLATIONS
(VALID ABOVE


CORRECTION OF MOMENT OF INERTIA
FOR TRANSLATIONAL OSCILLATIONS
VALID ABOVE

PHASE ANGLE FOR MOMENT OF INERTIA CORRECTION FOR TRANSLATIONAL OSCILLATIONS



PHASE ANGLE FOR MOMENT OF INERTIA CORRECTION FOR TRANSLATIONAL OSCILLATIONS

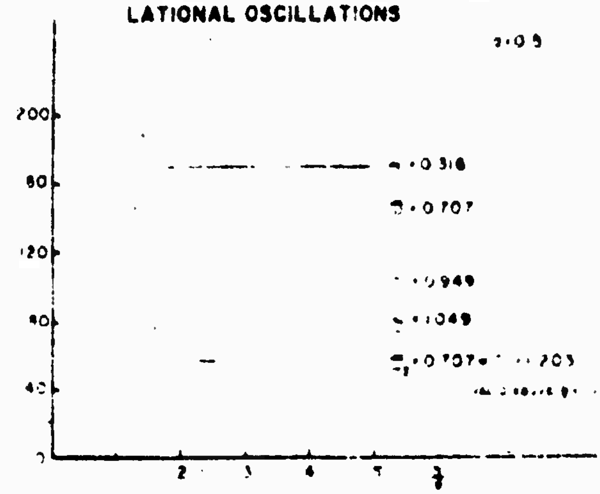
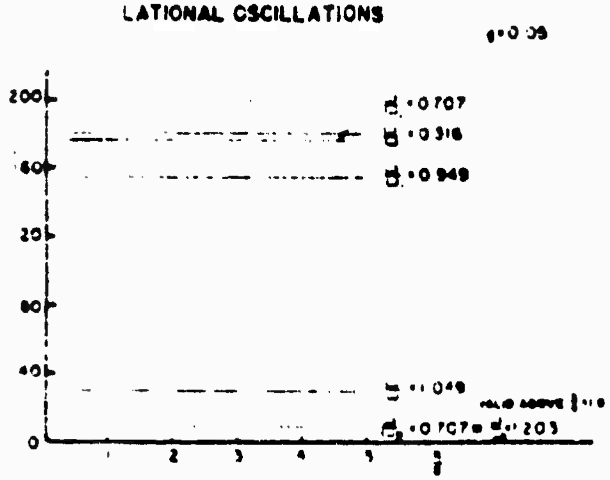
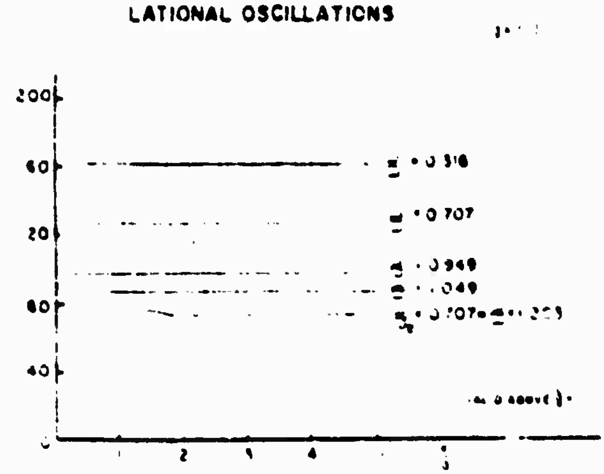


FIG. 5(b)

PHASE ANGLE FOR MOMENT OF INERTIA CORRECTION FOR TRANSLATIONAL OSCILLATIONS



PHASE ANGLE FOR MOMENT OF INERTIA CORRECTION FOR TRANSLATIONAL OSCILLATIONS



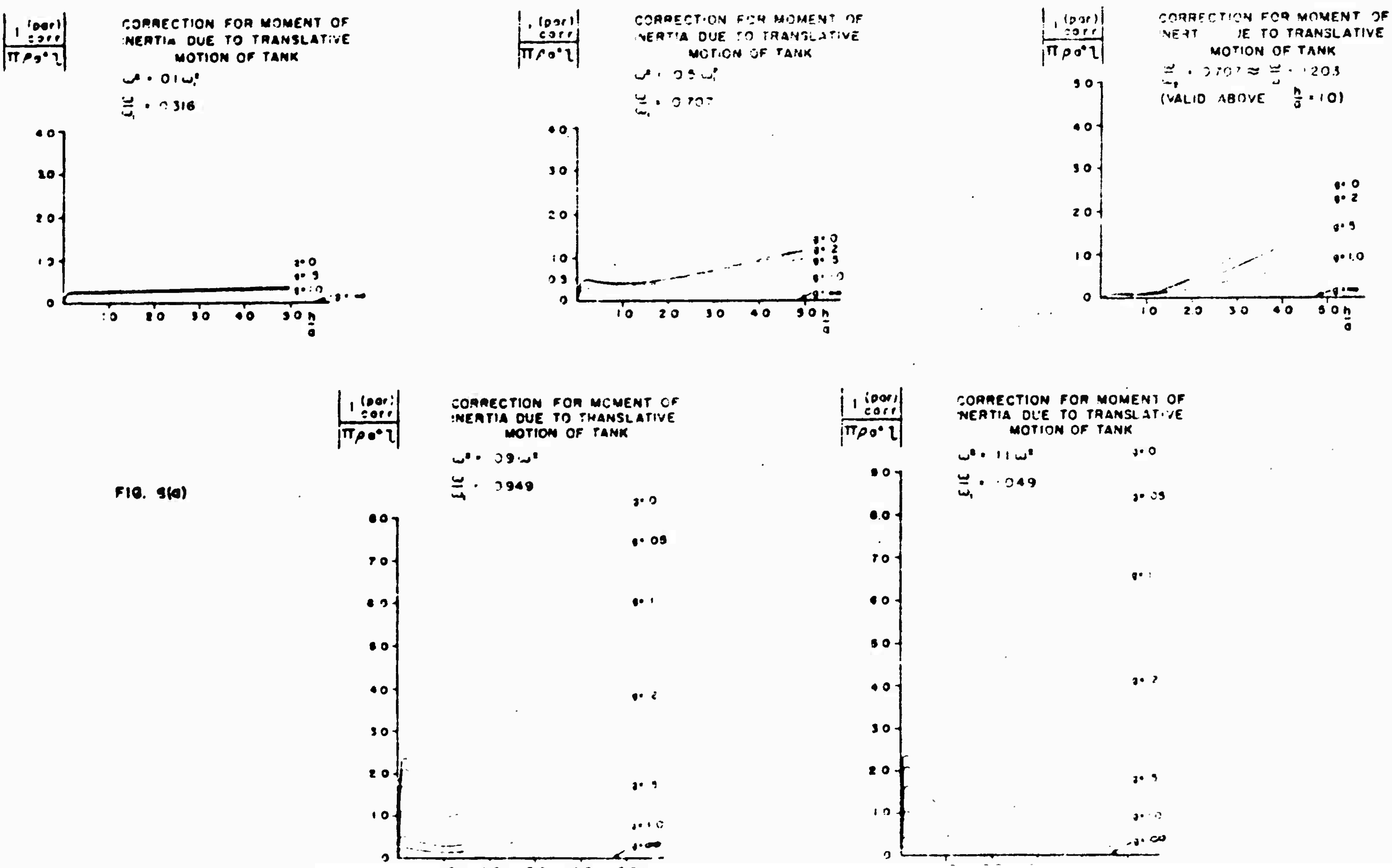
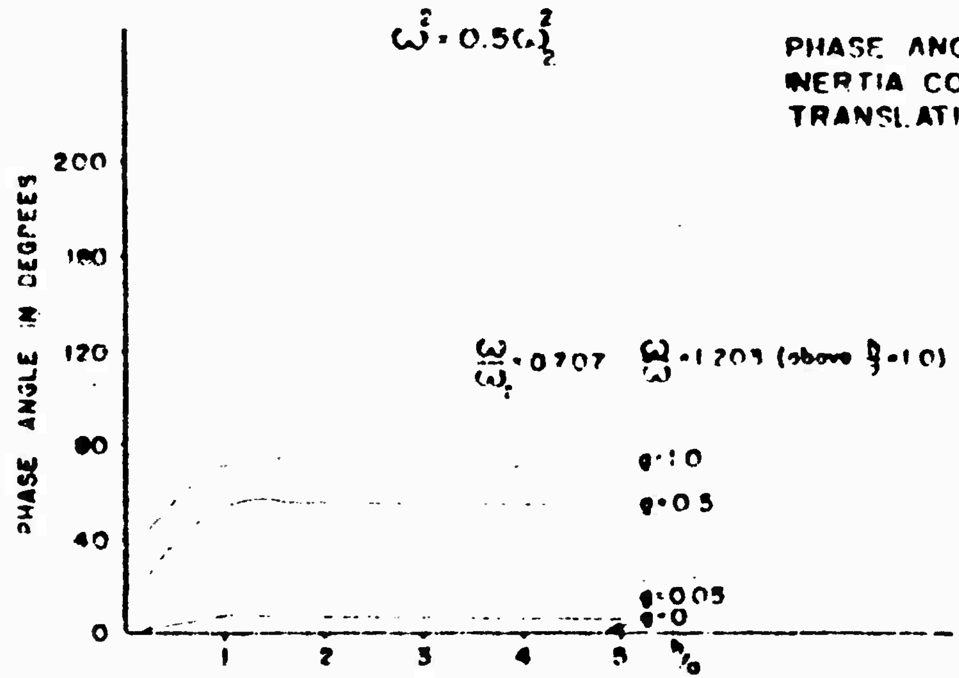
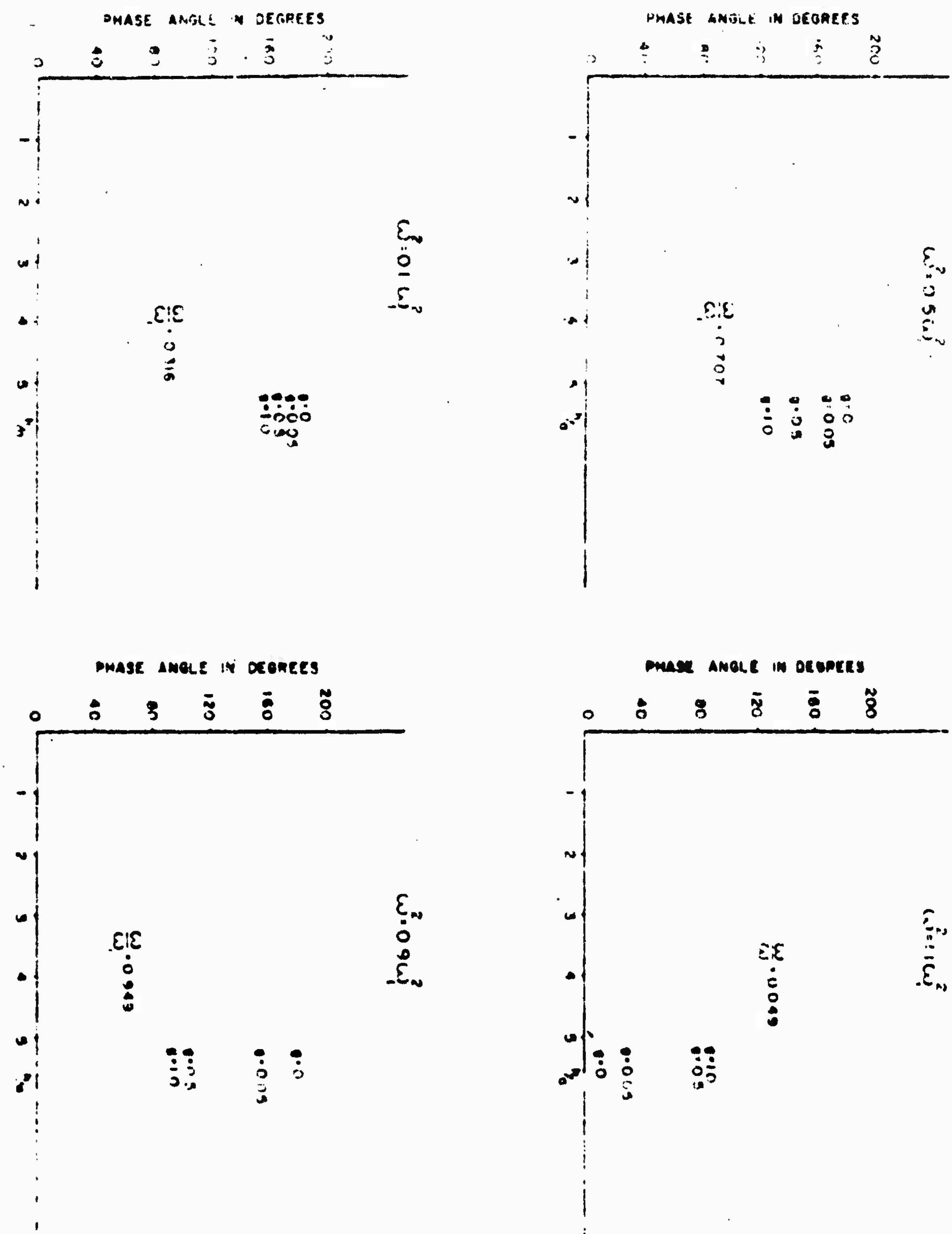

CORRECTION FOR MOMENT OF INERTIA DUE TO TRANSLATIVE MOTION OF TANK
CORRECTION FOR MOMENT OF INERTIA DUE TO TRANSLATIVE MOTION OF TANK
CORRECTION FOR MOMENT OF INERTIA DUE TO TRANSLATIVE MOTION OF TANK
(VALID ABOVE h/a = 10)
CORRECTION FOR MOMENT OF INERTIA DUE TO TRANSLATIVE MOTION OF TANK
CORRECTION FOR MOMENT OF INERTIA DUE TO TRANSLATIVE MOTION OF TANK

FIG. 9(a)



PHASE ANGLE FOR MOMENT OF INERTIA CORRECTION DUE TO TRANSLATIONAL TANK MOTION

FIG. 6(b)



PHASE ANGLE IN DEGREES
PHASE ANGLE IN DEGREES
PHASE ANGLE IN DEGREES
PHASE ANGLE IN DEGREES

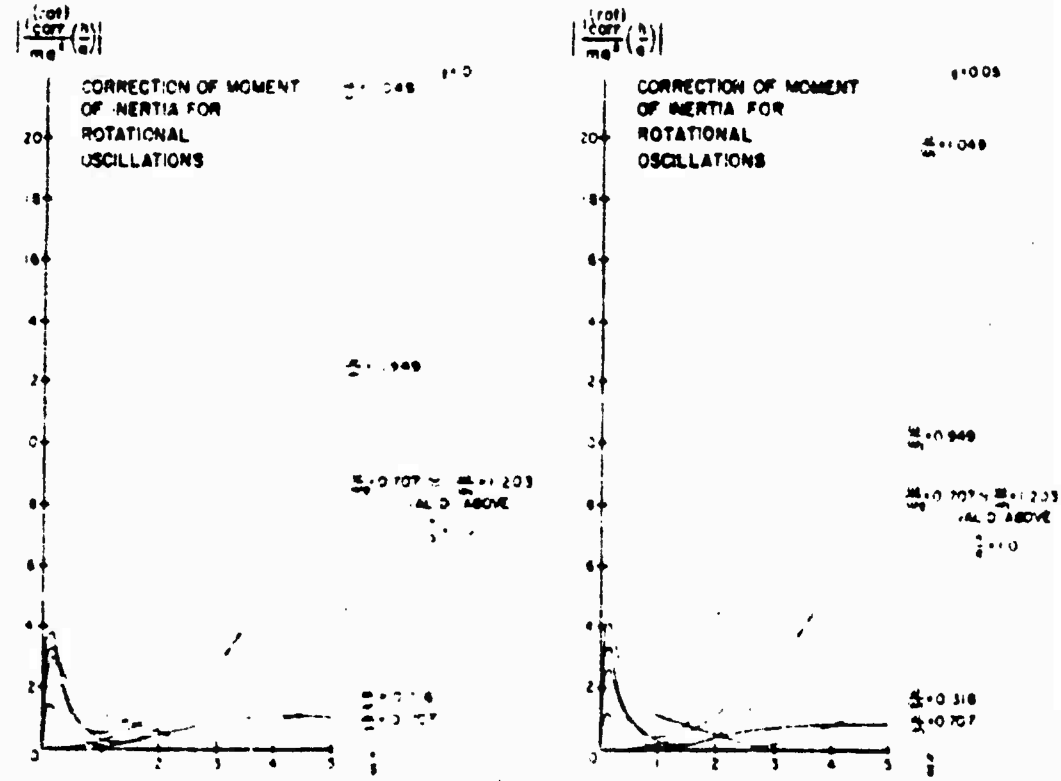
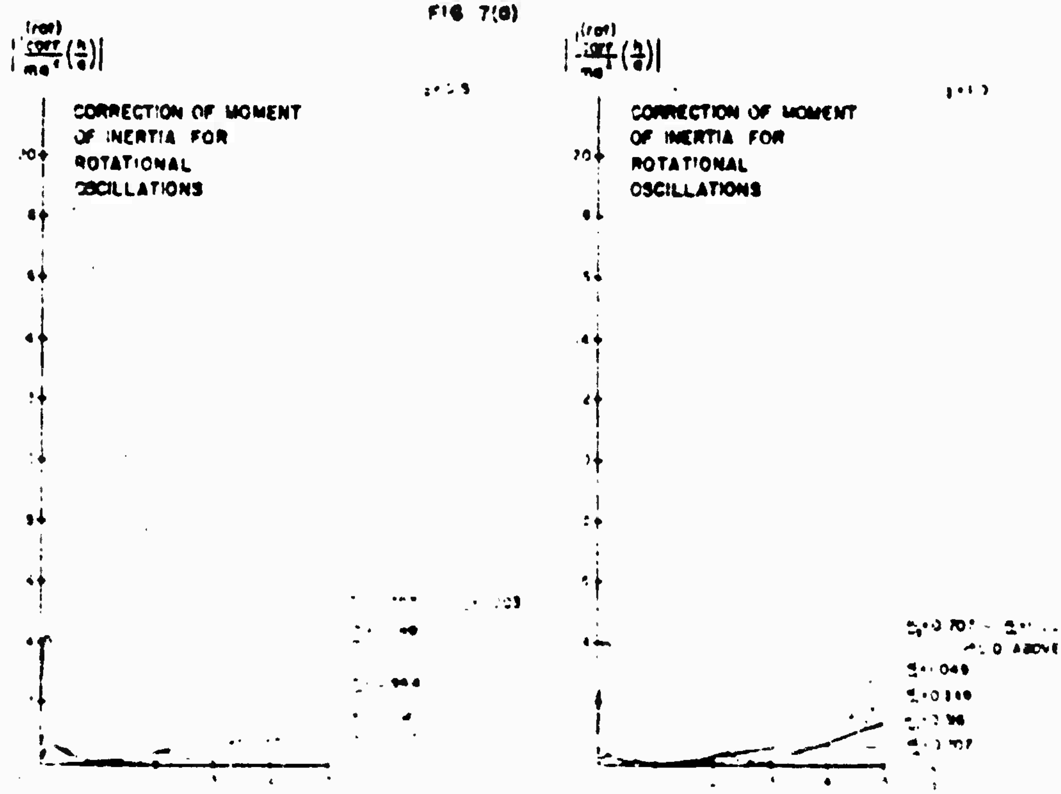
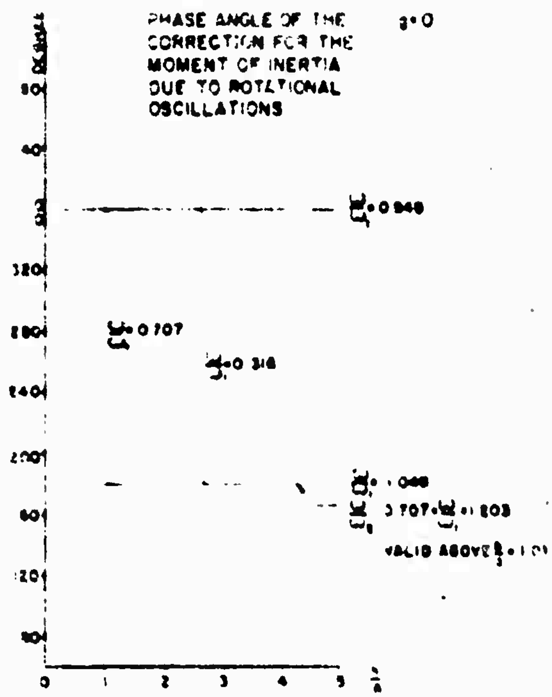

CORRECTION OF MOMENT OF INERTIA FOR ROTATIONAL OSCILLATIONS
CORRECTION OF MOMENT OF INERTIA FOR ROTATIONAL OSCILLATIONS
ALSO ABOVE

FIG 7(a)


CORRECTION OF MOMENT OF INERTIA FOR ROTATIONAL OSCILLATIONS
CORRECTION OF MOMENT OF INERTIA FOR ROTATIONAL OSCILLATIONS
ALSO ABOVE


PHASE ANGLE OF THE
CORRECTION FOR THE
MOMENT OF INERTIA
DUE TO ROTATIONAL
OSCILLATIONS
g = 0
DEGREES
(VALID ABOVE)

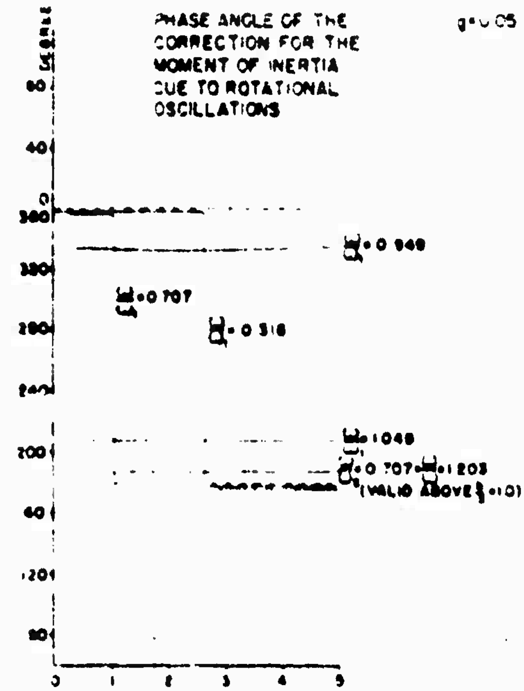
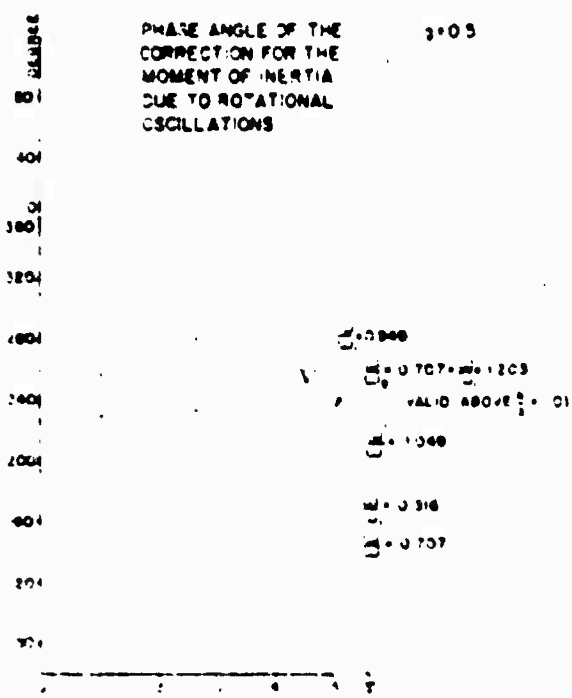
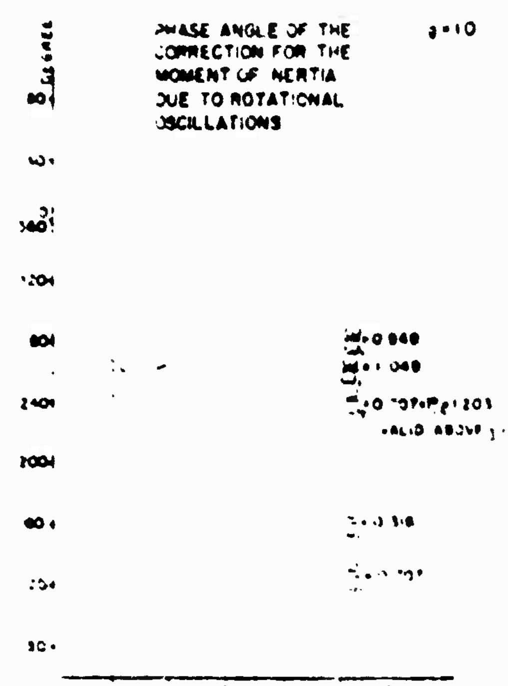

PHASE ANGLE OF THE
CORRECTION FOR THE
MOMENT OF INERTIA
DUE TO ROTATIONAL
OSCILLATIONS
g = 0.05
DEGREES
(VALID ABOVE)

FIG. 7(b)


PHASE ANGLE OF THE
CORRECTION FOR THE
MOMENT OF INERTIA
DUE TO ROTATIONAL
OSCILLATIONS
g = 0.5
DEGREES


PHASE ANGLE OF THE
CORRECTION FOR THE
MOMENT OF INERTIA
DUE TO ROTATIONAL
OSCILLATIONS
g = 1.0
DEGREES

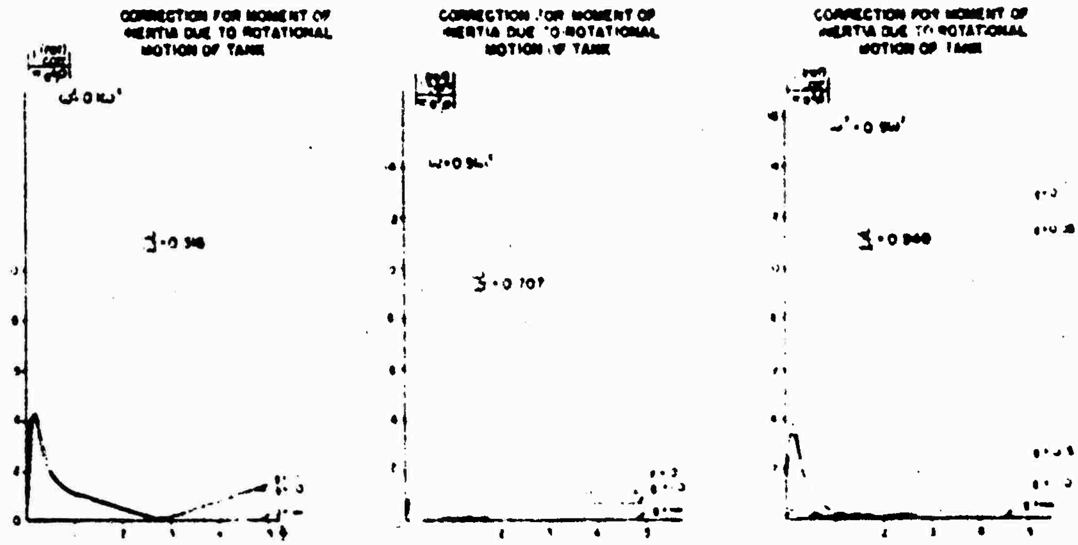
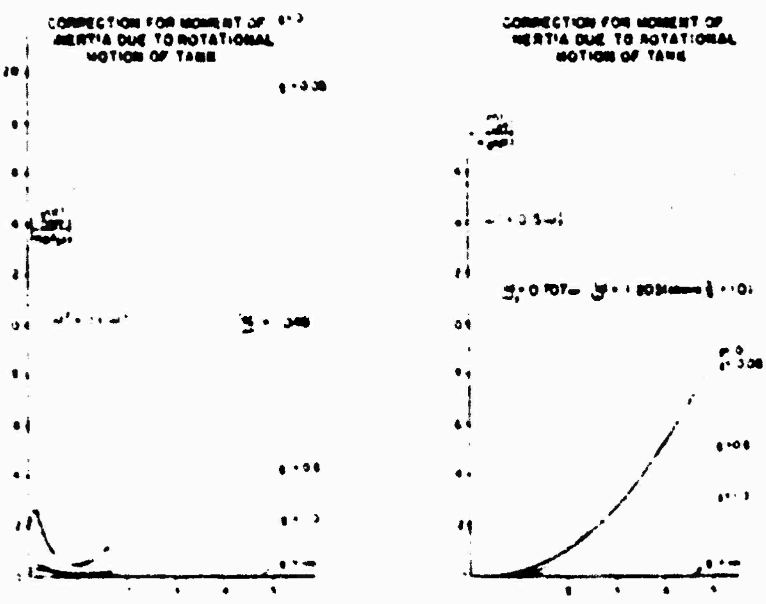
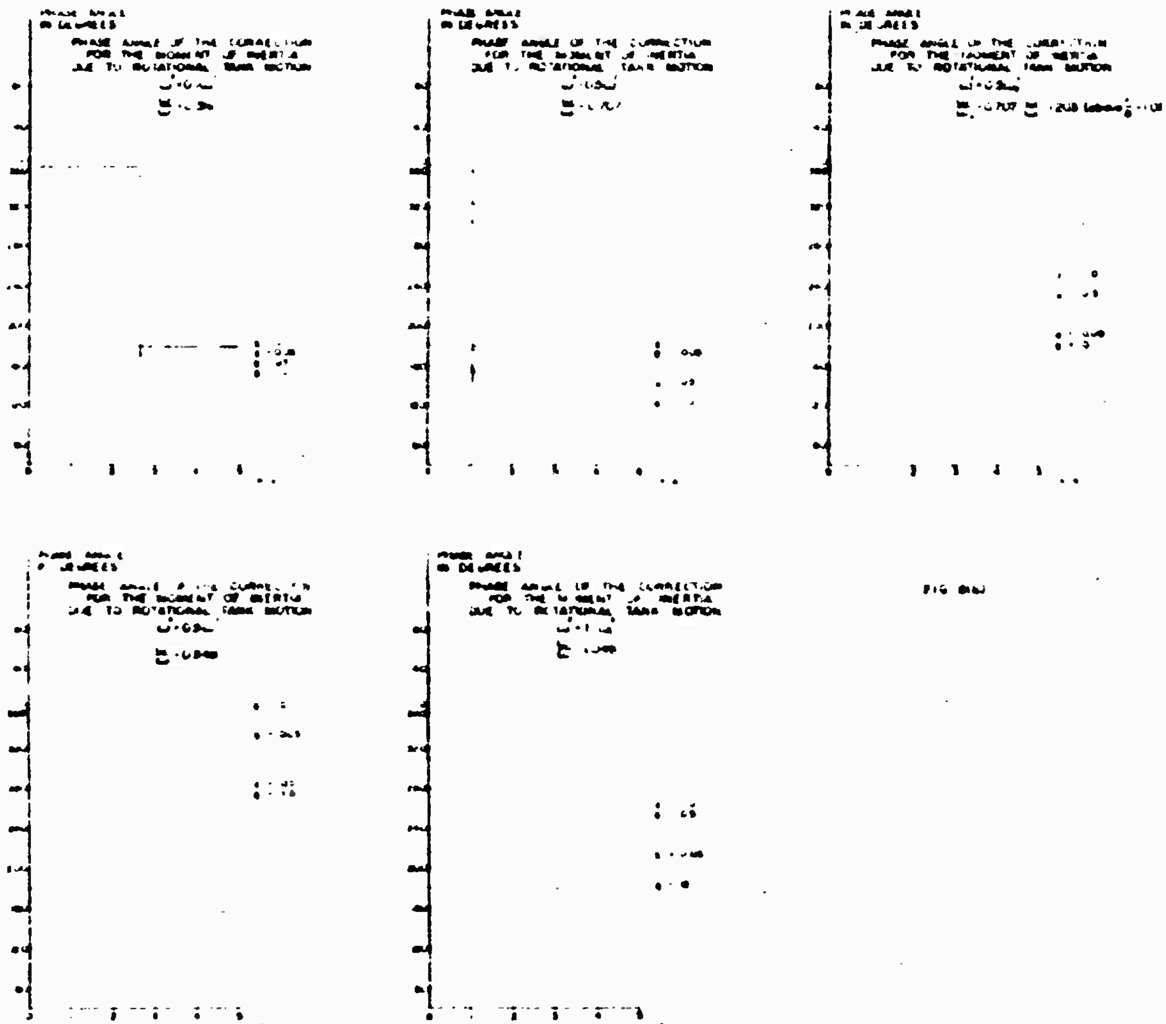

CORRECTION FOR MOMENT OF INERTIA DUE TO ROTATIONAL MOTION OF TANK
CORRECTION FOR MOMENT OF INERTIA DUE TO ROTATIONAL MOTION OF TANK
CORRECTION FOR MOMENT OF INERTIA DUE TO ROTATIONAL MOTION OF TANK

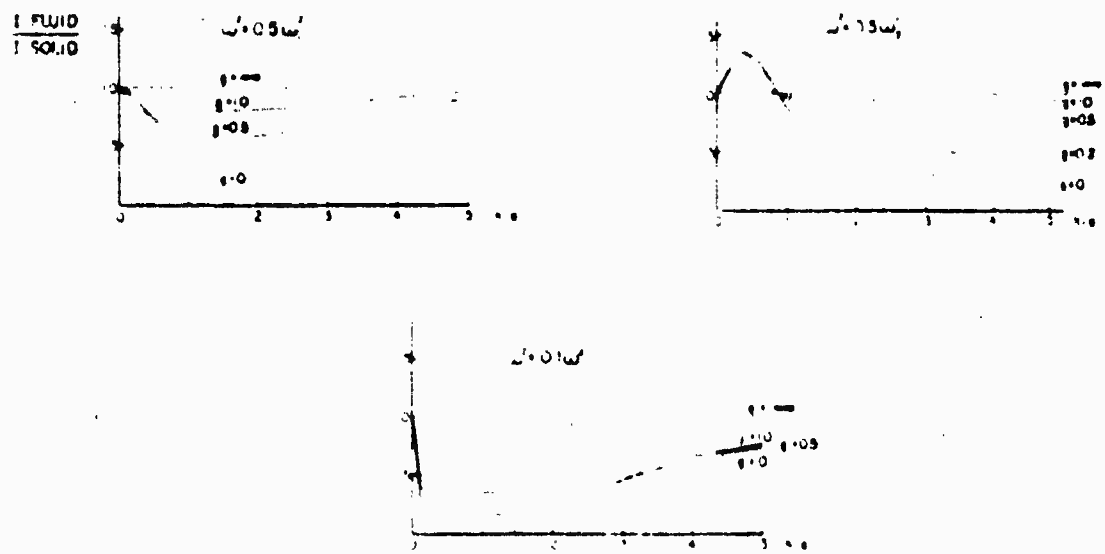
FIG 8(d)


CORRECTION FOR MOMENT OF INERTIA DUE TO ROTATIONAL MOTION OF TANK
CORRECTION FOR MOMENT OF INERTIA DUE TO ROTATIONAL MOTION OF TANK

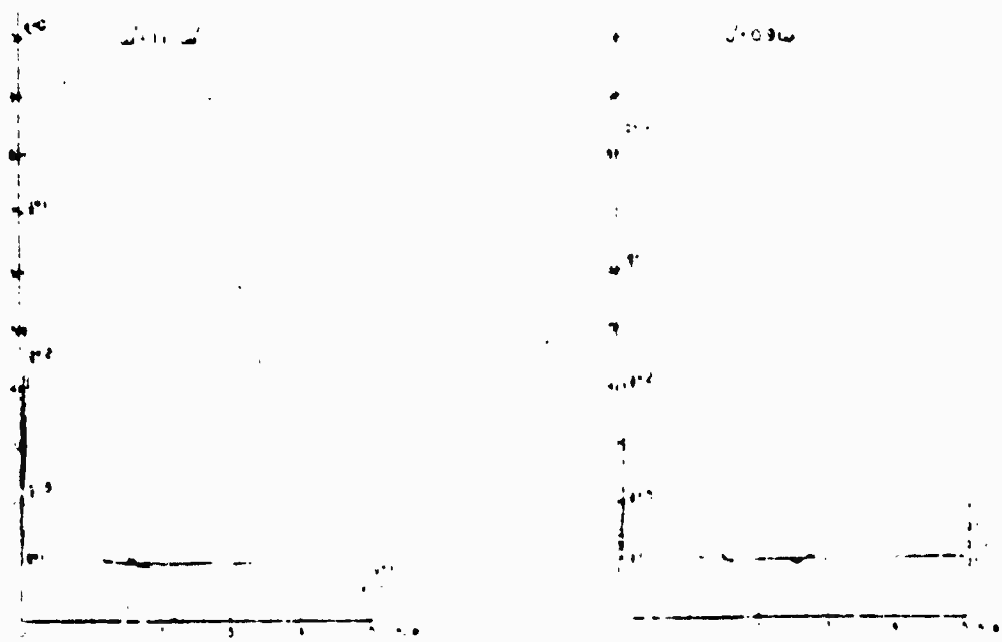


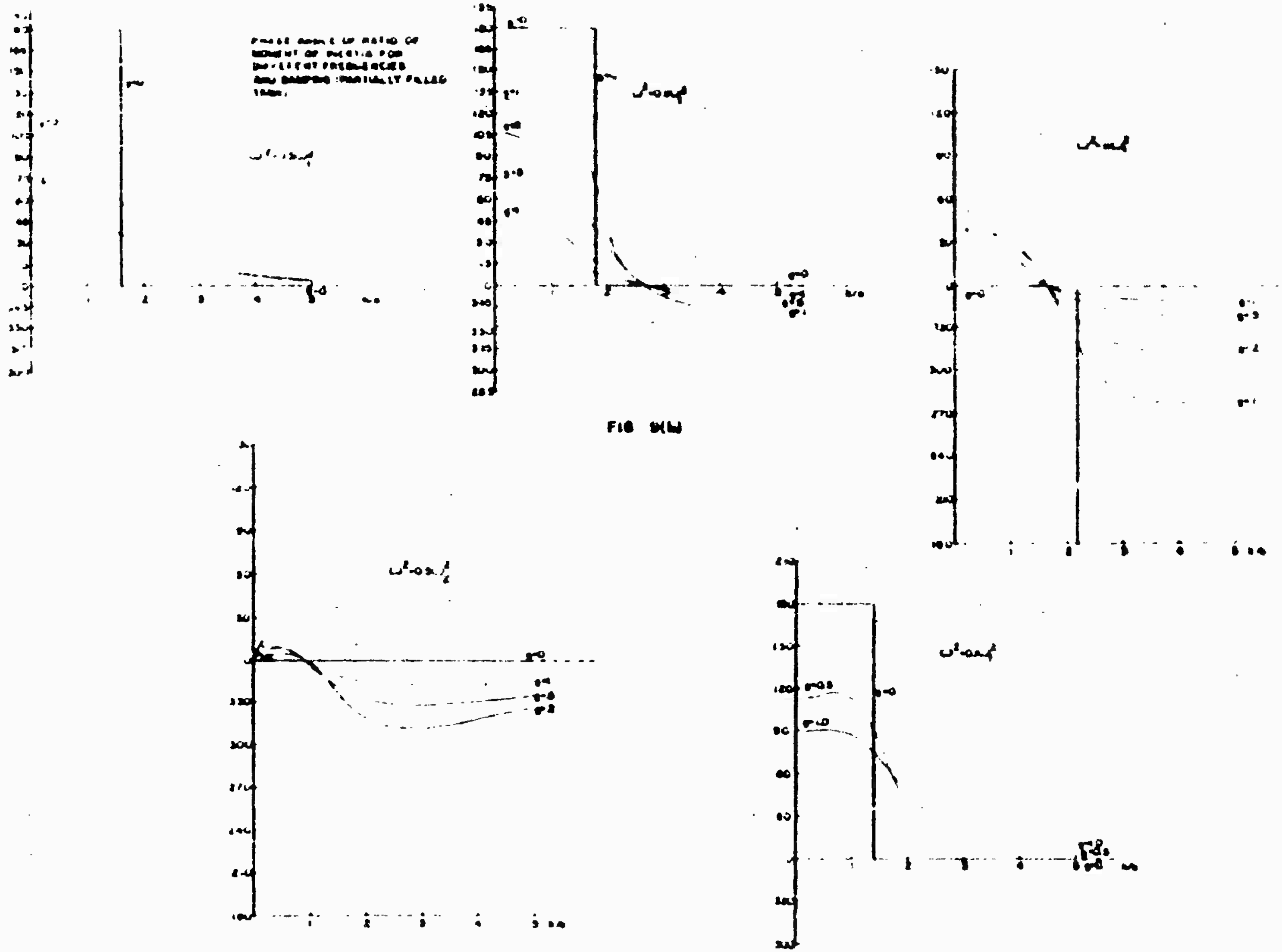
PHASE ANGLE IN DEGREES
PHASE ANGLE OF THE CORRECTION FOR THE MOMENT OF INERTIA DUE TO ROTATIONAL TANK MOTION

FIG. [illegible]



RATIO OF MOMENT OF INERTIA OF FLUID AND SOLID BODY VERSUS FLUID HEIGHT FOR DIFFERENT FREQUENCIES AND DAMPING (PARTIALLY FILLED TANK)   FIG 9(a)





PHASE ANGLE OF RATIO OF MOMENT OF INERTIA FOR DIFFERENT FREQUENCIES AND DAMPING (PARTIALLY FILLED TANK).

FIG 3(b)